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Nonlinear self-excited acoustic oscillations within fixed boundaries

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In §1 a brief discussion of the general problem of self-excited acoustic oscillations within fixed boundaries is given. In §2 a second-order analysis is developed for the special case of rectangular cavities. A nonlinear wave equation is derived for essentially arbitrary boundary conditions. The analysis can be extended to other cavity geometries provided that the first-order solutions can be expressed in closed form. Various applications of the analysis are discussed in §3. It turns out that two-dimensional problems of self-excited oscillations generally lead to nonlinear equations containing terms with a time lag. It is anticipated that the time lag (rather than viscous effects or sound radiation) represents the key to a fundamental understanding of the character of the oscillations and the variety of modes appearing in self-excited resonators.

1. Introduction

In recent years a substantial effort has been made to understand the physical mechanisms and geometric conditions that are responsible for the occurrence of self-excited acoustic oscillations of a gas contained in a cavity. Problems of this kind are difficult to investigate because the nonlinearity of the acoustic wave motion in the confined (or semiconfined) gas is always fundamentally important. In the absence of nonlinearity the corresponding wave equations would be *homogeneous*, in which case the amplitudes as well as the slopes of the solutions would remain undetermined. Indeed, the role of nonlinearity is more crucial for self-excited than for forced acoustic oscillations. To fix ideas we consider the two following equations for the acoustic quantity f (pressure, density, velocity etc.):

$$U = f\frac{df}{dt} + L_D f, \tag{1.1}$$

$$L_E f = f\frac{df}{dt} + L_D f, \tag{1.2}$$

where L_E and L_D are linear operators. Equation (1.1) has the typical form of a (second-order) *resonance equation*. The term U on the left-hand side may denote, for example, the periodic velocity of a piston at one end of a resonance tube; the expression $L_D f$ may account for effects due to the *Stokes boundary layer*, acoustic radiation, compressive viscosity, etc. The quadratic term in (1.1) stems from the second-order terms in the Navier–Stokes equations and is clearly important only if $L_D f$ is not too large. Hence, depending on the strength of the dissipative mechanisms, (1.1) may range from the limiting case where only the nonlinear term is important to the case of a linear equation. In the latter case the resonant amplitude is limited by the linear expression $L_D f$. A particularly interesting equation of the type (1.1) is Chester's resonance equation, which governs resonant acoustic oscillations in closed tubes and has been investigated in detail by Chester (1964) and Keller (1976*a*).

Equations of the form (1.2) are often found in connection with self-excited acoustic oscillations. In contrast with (1.1), where U is a given function of time, the excitation $L_E f$ in (1.2) is a linear expression in f. Hence in this case the dissipative mechanisms, accounted for by the linear expression $L_D f$, play an entirely different role to that in the case of forced oscillations. They are responsible for the stability of (1.2). The amplitudes of self-excited oscillations, on the other hand, are always limited by nonlinearity. Equations of this type have been studied by Chu (1963), Mitchell, Crocco and Sirignano (1969), Keller (1978*a*), and many others.

This rather simple but fundamental difference between forced and self-excited oscillations leads immediately to an interesting question. In the case of forced oscillations the energy addition to the confined gas is linear in f, dissipation and radiation of energy (as expressed by L_D) are quadratic in f, and shock-wave dissipation, which is included through the presence of the quadratic term in (1.1), is of third order. Thus it is possible that the energy addition is balanced by 'linear dissipation' only and that nonlinear effects are negligible. In this case a continuous excitation $U(t)$ leads to a continuous resonant response $f(t)$. In the case of self-excited oscillations of the type considered here, however, the occurrence of continuous solutions $f(t)$ is rather surprising. Considering that both $L_E f$ and $L_D f$ are linear in f, shock-wave dissipation, for example, would be an obvious mechanism capable of balancing the energy addition, which is *quadratic* in f.

On the other hand it is well known that self-excited oscillations are frequently close to sinusoidal or at least continuous. Consequently there must be further possibilities, through which amplitude nonlinearity (apparently in connection with certain properties of the linear terms $L_E f$ or $L_D f$) can act as an amplitude-limiting mechanism. Indeed, another effect capable of limiting amplitudes appears if we include suitable terms in $L_E f$ that contain one or more time lags (or even a convolution integral). Within the framework of a linear stability analysis, for example, the boundary conditions are usually considered to be quasistationary, i.e. inlet throttles, exit nozzles etc. are treated as acoustically compact elements if their dimensions are small compared with typical wavelengths produced by the resonator under consideration. Often such simplifications are not justified for second-order theories, as was indicated above. In general the time-dependent character of the boundary conditions together with nonlinearity incorporates a further mechanism of fundamental importance, which we might call 'self-detuning' of the resonator. The importance of the time lag for chemical reactions was pointed out first by Crocco & Cheng (1956), who developed a 'time-lag model', which has been used successfully by Mitchell *et al.* (1969) to derive a nonlinear equation for longitudinal self-excited oscillations in rocket motors with concentrated combustion. Their equation has the form

$$[f(t)-\lambda]\frac{df(t)}{dt} = af(t)+bf(t-t_0), \tag{1.3}$$

where λ, a and b are constants and t_0 is essentially the time lag. Although the time lag defined by Crocco & Cheng is associated with a characteristic time of combustion in the overall processes of burning liquid propellants, the idea can be used in a much more general way. If acoustic oscillations in a nearly closed cavity are induced owing to the presence of a flow outside the cavity or if, for instance, a thin wall-bounded jet enters a cavity, thus playing the role of a boundary element within the cavity (and possibly acting as a partner for a coupled oscillation), it is obvious that the convective character of the flow leads to time-lag effects. At the present time, however, the very difficult subclass of problems concerned with flow-induced vibrations will not be considered further. To formulate the basic ideas with respect to the problem of nonlinear acoustic oscillations within fixed boundaries we consider the problem of defining appropriate boundary conditions as a separate question. In order not to overstep the bounds of the present paper, but still to illustrate how certain boundary conditions lead to wave equations similar to (1.3), we consider a generalized version of Chu's (1963) problem in §3.3. In this case, however, the reason for the appearance of time-lag terms is not the time-dependent character of the boundary conditions, but a delay that is associated with the varying phase angles of incident and reflected waves along the cavity walls. The treatment of the nonlinear acoustical problem would in fact be the same for certain flow-induced vibrations, although an example of this kind would require a separate extensive consideration of the excitation mechanism to define the boundary conditions. It should be pointed out that in the case of flow-induced vibrations nonlinearity may also be introduced by effects accounted for by the boundary conditions.

Before we can proceed to develop a suitable analysis, the various kinds of self-excited acoustic oscillations must be graded appropriately. As the present problems are nonlinear we can only discuss certain subclasses of problems on the basis of a particular analysis. The most fundamental distinction arises from the choice of the boundary conditions. Here we restrict consideration to boundary conditions for the particle velocity alone. Thus it is justified to express all quantities as expansions in a typical Mach number. If pressure boundary conditions were also included, it would be necessary to introduce two separate expansion schemes for the travelling times of waves and their amplitudes (see Keller 1977) in order to account for lowest-order amplitude dispersion. Furthermore, we assume that the maximum pressure disturbances are small compared with the mean pressure. Energy addition, dissipation and radiation are considered to be second- (and higher-) order effects only.

The problem then consists of two main parts. We have first to find the general second-order solution and secondly to impose the velocity boundary conditions to second order. These two steps will lead to an equation that governs the nonlinear wave motion and for which time is the only independent variable. For simplification we assume that the attenuation of acoustic waves due to viscous effects is negligibly small.

Whilst for one-dimensional problems it is relatively easy to adjust second-order solutions to arbitrary boundary conditions, this represents the main difficulty of two- and three-dimensional problems. It can be overcome by replacing the actual boundary conditions by certain simplified boundary conditions that produce the same first-order wavefield. Such a replacement is possible because the boundary conditions that belong to a given first-order wavefield are not unique.

2. Analysis

As the present problem is nonlinear, and consequently solutions cannot be superposed, we must restrict our considerations to a special cavity geometry from the beginning of the analysis. Although the subsequent second-order analysis is presently restricted to rectangular cavities, it can be extended to more general geometries provided that the general first-order wavefield can be expressed in closed form.

As the second-order velocity components for resonant oscillations have already been derived by Keller (1978*b*) we can simply state the main results and then proceed to formulate the methods to solve the more difficult problem of self-excited oscillations.

If the corners of a rectangular cavity are defined by

$$(x,\, y) = (0,\, 0),\ (a,\, 0),\ (0,\, b),\ (a,\, b), \tag{2.1}$$

the possible eigenfrequencies are

$$\nu_{m,\,n} = \frac{1}{2\pi}\omega_{m,\,n} = \tfrac{1}{2}c_0\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^{\frac{1}{2}}, \tag{2.2}$$

where c_0 is the undisturbed sound speed, and m and n are non-negative integers, at least one of which is different from zero. For each of these eigenfrequencies there is a mode produced by superposition of the four waves

$$f^{(m,\,n)}_{\pm\pm} = f\left(t \pm \frac{x}{c_0}\cos\phi_{m,\,n} \pm \frac{y}{c_0}\sin\phi_{m,\,n}\right), \tag{2.3}$$

where
$$\phi_{m,\,n} = \arctan\frac{a}{b}\frac{n}{m}.$$

We use the small amplitude expansions

$$\left.\begin{aligned} u &= \epsilon u_1 + \epsilon^2 u_2 + O(\epsilon^3),\\ v &= \epsilon v_1 + \epsilon^2 v_2 + O(\epsilon^3),\\ c_0 + c &= c_0 + \epsilon c_1 + \epsilon^2 c_2 + O(\epsilon^3) \end{aligned}\right\} \tag{2.4}$$

for the x-component u and the y-component v of the particle velocity and the sound speed c_0+c. The expansion parameter ϵ can be interpreted as the square root of the maximum Mach number M_{w} at the wall (based upon the velocity components normal to the wall):

$$\epsilon = M_{\mathrm{w}}^{\frac{1}{2}}. \tag{2.5}$$

For the special case of a piston-driven resonance tube M_{w} is the maximum piston velocity divided by the mean sound speed in the gas column that is contained in the tube (see e.g. Chester 1964).

The first-order quantities can then be expressed as follows:

$$\left.\begin{aligned} \epsilon u_1 &= \tfrac{1}{4}\cos\phi\,\{f_{--}-f_{++}+f_{-+}-f_{+-}\},\\ \epsilon v_1 &= \tfrac{1}{4}\sin\phi\,\{f_{--}-f_{++}-f_{-+}+f_{+-}\},\\ \epsilon c_1 &= \tfrac{1}{8}(\gamma-1)\{f_{--}+f_{++}+f_{-+}+f_{+-}\}, \end{aligned}\right\} \tag{2.6}$$

where γ is the ratio of specific heats and, for convenience, the superscript $(m,\, n)$ has been dropped. These first approximations can now be inserted in the second-order terms of the Eulerian equations and a second approximation found by iteration. This has been discussed before by Keller (1978*b*) and is not repeated here, although it should be pointed out that for reasons of symmetry the signs in the definition (2.3) have been changed. The iteration is straightforward, though tedious, and leads after integration to the results

$$\begin{aligned} \epsilon^2 u_2 = \frac{\gamma+1}{64c_0^2}\cos\phi\Big\{&[(x-a)\cos\phi+(y-b)\sin\phi]\frac{\partial}{\partial t}[f_{--}^2+f_{++}^2]\\ &+[(x-a)\cos\phi-(y-b)\sin\phi]\frac{\partial}{\partial t}[f_{-+}^2+f_{+-}^2]\Big\}+A(x,\, y,\, t), \end{aligned} \tag{2.7}$$

$$\epsilon^2 v_2 = \frac{\gamma+1}{64c_0^2}\sin\phi\left\{[(x-a)\cos\phi+(y-b)\sin\phi]\frac{\partial}{\partial t}[f_{--}^2+f_{++}^2]\right.$$
$$\left.-[(x-a)\cos\phi-(y-b)\sin\phi]\frac{\partial}{\partial t}[f_{-+}^2+f_{+-}^2]\right\}+B(x,y,t) \qquad (2.8)$$

for $(0<\phi<\frac{1}{2}\pi)$.

The functions A and B can be considered to be sums

$$A = A_{\mathrm{I}}+A_{\mathrm{II}}, \quad B = B_{\mathrm{I}}+B_{\mathrm{II}}, \qquad (2.9)$$

where A_{I} and B_{I} account for the interaction of waves in the cavity, and disappear identically at the walls (see corresponding remarks by Keller 1978*b*). The terms A_{II} and B_{II} are functions of integration, i.e. eigenfunctions of the linearized Eulerian equations, and have the general forms

$$A_{\mathrm{II}}(x,y,t) = \int_0^{2\pi}\psi\left(t-\frac{x}{c_0}\cos\alpha-\frac{y}{c_0}\sin\alpha,\alpha\right)\cos\alpha d\alpha, \qquad (2.10)$$

$$B_{\mathrm{II}}(x,y,t) = \int_0^{2\pi}\psi\left(t-\frac{x}{c_0}\cos\alpha-\frac{y}{c_0}\sin\alpha,\alpha\right)\sin\alpha d\alpha. \qquad (2.11)$$

With this it is easy to show that an arbitrary boundary condition at the walls can be satisfied by $\epsilon u_1+\epsilon^2 u_2$ and $\epsilon v_1+\epsilon^2 v_2$, as defined by (2.6), (2.7) and (2.8). Knowing the boundary conditions, the remaining problem would be to evaluate expressions (2.10) and (2.11). Transformations of second- and third-order equations generated by such eigenfunctions may be usefully employed to rewrite in a more suitable form (see Keller 1976*b*) nonlinear equations for one-dimensional wave-propagation problems (e.g. the periodic shock-tube problem). For two- or three-dimensional wave-propagation problems, however, such an analysis would require a great deal of unnecessary algebra.

The present procedure, chosen to adapt the boundary conditions, is equivalent to adjusting the eigenfunctions but requires less algebra and is considerably more transparent. It has been applied previously by Keller (1978*b*) for sinusoidal excitation at the walls of a rectangular cavity, and is extended here to arbitrary excitation functions. The idea is to define a suitable orthogonal projection that acts as a 'resonance filter' when applied to the boundary conditions. Without essential loss of generality, we define the boundary conditions by

$$\left.\begin{aligned} u(a,y,t)=0, \quad v(x,b,t)=0,\\ u(0,y,t)=u_{\mathrm{w}}(y,t), \quad v(x,0,t)=v_{\mathrm{w}}(x,t).\end{aligned}\right\} \qquad (2.12)$$

Now we postulate that an othogonal projection

$$P[(u_{\mathrm{w}},v_{\mathrm{w}})] = (u_{\mathrm{wR}},v_{\mathrm{wR}}) \qquad (2.13)$$

can be defined as follows. Let the velocities u_{w} and v_{w} at the walls be split into a pair of 'resonant excitation functions' u_{wR} and v_{wR} and a pair of 'non-resonant excitation functions' u_{wA} and v_{wA},

$$u_{\mathrm{w}} = u_{\mathrm{wR}}+u_{\mathrm{wA}}, \quad v_{\mathrm{w}} = v_{\mathrm{wR}}+v_{\mathrm{wA}}, \qquad (2.14)$$

such that u_{wR} *and* v_{wR} *produce a uniform rate of energy addition along the wavefronts* and the mean energy addition produced by u_{wA} and v_{wA} is zero. It will be shown later that the *reduced* boundary conditions $(u_{\mathrm{wR}},v_{\mathrm{wR}})$ can be satisfied by the second-order components of the particle velocity defined by (2.7) and (2.8) without using the functions A and B, i.e. the eigenfunctions A_{II} and B_{II} become trivial: $A_{\mathrm{II}}\equiv B_{\mathrm{II}}\equiv 0$.

Obviously (2.13) implies

$$P[(u_{\mathrm{wA}}, v_{\mathrm{wA}})] = (0, 0),$$
$$P[(u_{\mathrm{wR}}, v_{\mathrm{wR}})] = (u_{\mathrm{wR}}, v_{\mathrm{wR}}).$$

As an example we consider the special case defined by

$$b = \tfrac{1}{2}a, \quad u_{\mathrm{w}}(y, t) = \epsilon_1 c_0 \sin \omega t, \quad v_{\mathrm{w}}(x, t) = \epsilon_2 c_0 \sin (\omega t - \theta),$$

where $\omega a = c_0 \pi$. In this case u_{w} produces a resonant response of the wavefield in the cavity, whereas v_{w} causes a linear response only. In other words, only a represents a resonant length. Consequently, we find

$$(u_{\mathrm{wR}}, v_{\mathrm{wR}}) = (\epsilon_1 c_0 \sin \omega t, 0),$$
$$(u_{\mathrm{wA}}, v_{\mathrm{wA}}) = (0, \epsilon_2 c_0 \sin (\omega t - \theta)),$$

and therefore

$$P[(u_{\mathrm{w}}, v_{\mathrm{w}})] = (u_{\mathrm{w}}, 0).$$

In this simple case it is obvious that u_{wR} and v_{wR} provide a uniform rate of energy addition along the wavefronts. Here it should be pointed out that the present analysis is restricted to cases where the amplitudes of non-resonant-excitation functions are of the same order of magnitude (or smaller) as the amplitudes of the resonant excitation functions. For the previous example this implies

$$O(\epsilon_1) \geqq O(\epsilon_2).$$

When the solutions for c, for example, are written in the form (see (2.4) and (2.5))

$$c(x, y, t) = M_{\mathrm{w}}^{\frac{1}{2}} c_1(x, y, t) + M_{\mathrm{w}} c_2(x, y, t) + O(M_{\mathrm{w}}^{\frac{3}{2}}), \tag{2.15}$$

c_1 is determined by resonant excitation functions. The lowest-order effects of non-resonant excitation functions are accounted for by c_2. Hence, if we are only interested in the lowest-order part $M_{\mathrm{w}}^{\frac{1}{2}} c_1$ of the solution c, it is justified to 'eliminate' non-resonant excitation functions $O(M_{\mathrm{w}})$.

The computation of the projected components u_{wR} and v_{wR} for arbitrary excitation functions u_{w}, v_{w} according to the definition of P is not straightforward but involves rather complicated arguments based upon a geometric consideration. To avoid a time-consuming derivation we simply state the results and prove their correctness. If the boundary conditions u_{w} and v_{w} are defined by (2.12), their projections u_{wR} and v_{wR} are given by

$$u_{\mathrm{wR}}(y, t) = \frac{\cos^2 \phi}{2b} \sum_{k=\pm 1} \sum_{l=\pm 1} \left\{ \int_0^b u_{\mathrm{w}}\left(\eta, t + \frac{k\eta + ly}{c_0} \sin \phi\right) d\eta + \int_0^a v_{\mathrm{w}}\left(\xi, t + k\frac{\xi}{c_0} \cos \phi + l\frac{y}{c_0} \sin \phi\right) d\xi \right\}, \tag{2.16}$$

$$v_{\mathrm{wR}}(x, t) = \frac{\sin^2 \phi}{2a} \sum_{k=\pm 1} \sum_{l=\pm 1} \left\{ \int_0^b u_{\mathrm{w}}\left(\eta, t + k\frac{x}{c_0} \cos \phi + l\frac{\eta}{c_0} \sin \phi\right) d\eta + \int_0^a v_{\mathrm{w}}\left(\xi, t + \frac{k\xi + l\xi}{c_0} \cos \phi\right) d\xi \right\}, \tag{2.17}$$

where

$$0 < \phi < \tfrac{1}{2}\pi. \tag{2.18}$$

Note that the wave-propagation angle ϕ should not assume the limiting values 0 or

$\frac{1}{2}\pi$ before the integrations in (2.16) and (2.17) are carried out, otherwise u_{wR} and v_{wR} would be incorrect by a factor of 2, as (2.16) and (2.17) incorporate Fourier integrals that become trivial in both limits.

Having postulated that (2.16) and (2.17) represent the resonant excitation functions u_{wR} and v_{wR} the proof can be given *a posteriori* by showing first that $u_w - u_{wR}$ and $v_w - v_{wR}$ are non-resonant excitation functions (see §A1 of the appendix), and secondly that the boundary conditions (2.16) and (2.17) can be satisfied by the reduced second-order velocity components (2.7) and (2.8) (i.e. $A_{II} \equiv B_{II} \equiv 0$) together with small first-order corrections that account for the possibility of small deviations of the frequencies from the resonant frequencies:

$$-\tfrac{1}{2}\lambda a \cos^2\phi\left\{f'\left(t-\frac{y}{c_0}\sin\phi\right)+f'\left(t+\frac{y}{c_0}\sin\phi\right)\right\}+\epsilon^3 u_2(0,y,t) = u_{wR}(y,t), \quad (2.19)$$

$$-\tfrac{1}{2}\lambda b \sin^2\phi\left\{f'\left(t-\frac{x}{c_0}\cos\phi\right)+f'\left(t+\frac{x}{c_0}\cos\phi\right)\right\}+\epsilon^2 v_2(x,0,t) = v_{wR}(x,t). \quad (2.20)$$

The first-order correction terms in (2.19) and (2.20) are obtained by inserting the boundary conditions $\epsilon u_1(a,y,t) = 0$ and $\epsilon v_1(x,b,t) = 0$ in the first-order expressions (2.6) for the velocity components and assuming a small deviation $\Delta\omega = \lambda c_0 \omega_0$ from the resonant angular frequency ω_0. To ensure second-order smallness of the corrections we require (in agreement with the usual restrictions, see e.g. Chester 1964) that $|\Delta\omega/\omega_0|$ should not be larger than $O(M_w^{\frac{1}{2}})$.

Attention is drawn to the fact that the terms $\epsilon^2 u_2(0,y,t)$ and $\epsilon^2 v_2(x,0,t)$ and the first-order corrections have the same argument structure as (2.16) and (2.17) (see §A2 of the appendix). This is the key that enables us to eliminate the spatial variables and to reduce the second-order partial differential equations to an ordinary nonlinear equation. Thus introducing (2.16) and (2.17) in (2.19) and (2.20) leads with the help of (2.7) and (2.8) to

$$\lambda\frac{df}{dt}+\frac{\gamma+1}{8c_0^2}f\frac{df}{dt}+\frac{1}{ab}\left\{\int_0^b\left[u_w\left(\eta,t+\frac{\eta}{c_0}\sin\phi\right)+u_w\left(\eta,t-\frac{\eta}{c_0}\sin\phi\right)\right]d\eta\right.$$
$$\left.+\int_0^a\left[v_w\left(\xi,t+\frac{\xi}{c_0}\cos\phi\right)+v_w\left(\xi,t-\frac{\xi}{c_0}\cos\phi\right)\right]d\xi\right\} = 0, \quad (2.21)$$

where $0 < \phi < \frac{1}{2}\pi$. This equation represents the principal result of the present analysis.

After defining the boundary conditions $u_w(y,t)$ and $v_w(x,t)$, the remaining problem is to find solutions of (2.21). Here it should be pointed out that correct limits of (2.21) cannot be expected when ϕ is set equal to one of the limiting values 0 or $\frac{1}{2}\pi$ (consider, for instance, the wavefronts of a genuine two-dimensional mode in a very long and narrow rectangular resonator). Nevertheless, the form of the wave equation remains unchanged in the two limits. To obtain the correct one-dimensional equations, when ϕ is set equal to 0 or $\frac{1}{2}\pi$, the term within the curly brackets must be divided by 2 before the integrations are carried out (see also the corresponding remarks below (2.18)). Furthermore, the nonlinear term is to be multiplied by 2 when ϕ assumes one of the limiting values. The fact that the coefficient of the term within the curly brackets becomes incorrect in the two limits of ϕ is an obvious outcome of the theory of Fourier integrals. Less transparent are the incorrect limits of the nonlinear term, although (2.21) is clearly compatible with the energy equation, as has been shown by Keller (1978*b*) for forced oscillations. However, the physical reason for this factor 2 is remarkably simple: for all genuine two-dimensional modes, the total length of the wavefront multiplied by the wavelength is equal to four times the resonator area. For one-dimensional modes this product is equal to twice the resonator area only. Hence, if we compare, for example, shock waves of equal magnitude, we find that the energy dissipated per cycle for genuine two-dimensional wavefields is twice as large as for one-dimensional wavefields.

3. Cavity oscillations

The aim is now to apply the concepts of §2 to the calculation of nonlinear oscillations in rectangular cavities.

3.1. *One-dimensional oscillations*

To recover the one-dimensional wave equation, special versions of which were first derived by Chu (1963) and Chester (1964), we set the mode angle ϕ equal to zero, divide the term within the curly brackets of (2.21) by two, and multiply the nonlinear term by two (see the remarks below (2.21)). Thus the integrations become trivial and we obtain

$$\lambda f'(t)+\frac{\gamma+1}{4c_0^2}f'(t)f(t)+\frac{1}{a}u_w(t)=0. \tag{3.1}$$

3.2. *Forced resonant oscillations in rectangular cavities*

To illustrate the special case of forced resonant oscillations in cavities we choose a wall-displacement motion of the form

$$u_w(y,t)=A(y)\cos\omega t+B(y)\sin\omega t, \tag{3.2}$$

$$v_w(x,t)=C(x)\cos\omega t+D(x)\sin\omega t, \tag{3.3}$$

and exclude the trivial cases $\phi=0$ and $\phi=\frac{1}{2}\pi$. Here ω is assumed to be within a neighbourhood

$$|(\omega-\omega_0)/\omega_0|\leqslant O(\epsilon) \tag{3.4}$$

of a certain resonant frequency ω_0 defined by the relation (2.2). Thus we have

$$\frac{\omega_0\cos\phi}{c_0}=\frac{m\pi}{a},\quad \frac{\omega_0\sin\phi}{c_0}=\frac{n\pi}{b}, \tag{3.5}$$

with $m\neq 0$ and $n\neq 0$.

Introducing the expressions (3.2) and (3.3) in the wave equation (2.21) leads, with the help of (3.5), to

$$\lambda f'(t)+\frac{\gamma+1}{8c_0^2}f'(t)f(t)+\left[\frac{A_n}{a}+\frac{C_m}{b}\right]\cos\omega t+\left[\frac{B_n}{a}+\frac{D_m}{b}\right]\sin\omega t=0, \tag{3.6}$$

where A_n, B_n, C_m and D_m denote Fourier coefficients and are given by

$$A_n=\frac{2}{b}\int_0^b A(\eta)\cos\left(n\pi\frac{\eta}{b}\right)d\eta,\quad B_n=\frac{2}{b}\int_0^b B(\eta)\cos\left(n\pi\frac{\eta}{b}\right)d\eta, \tag{3.7}$$

$$C_m=\frac{2}{a}\int_0^a C(\xi)\cos\left(m\pi\frac{\xi}{a}\right)d\xi,\quad D_m=\frac{2}{a}\int_0^a D(\xi)\cos\left(m\pi\frac{\xi}{a}\right)d\xi. \tag{3.8}$$

Making use of the linear theory λ is easily identified as

$$\lambda = \frac{\omega - \omega_0}{\omega_0 c_0}. \tag{3.9}$$

The wave equation (3.6) may again be recognized as Chester's equation. It should be pointed out that the present problem represents an extension of the corresponding problem that was discussed previously by Keller (1978*b*), where the wall-displacement motion was restricted to standing waves. However, (2.21) permits consideration of arbitrary excitations.

3.3. *Self-excited oscillations in rectangular cavities*

Clearly the most interesting applications of (2.21) concern the problems associated with nonlinear self-excited oscillations. In particular various jet-driven oscillations in cavities can be discussed on the basis of the wave equation (2.21). Typical equations obtained from (2.21), after inserting the boundary conditions, are of the form (1.3) but often contain the additional time-lag term $f'(t-t_0)$, i.e.

$$[f(t)-\lambda]\frac{df(t)}{dt} = af(t)+bf(t-t_0)+e\frac{df(t-t_0)}{dt}. \tag{3.10}$$

An interesting difference, however, is that equations of the type (3.10) (in contrast to those of the type (1.3)) do not admit discontinuous solutions (i.e. shock waves). Hereby the time lag t_0 appears as a natural consequence of convection and is directly related to the jet speed. Often such non-linear wave equations contain more than one time lag.

Here a somewhat simpler example is used to illustrate the time-lag character of self-excited oscillations. Following Chu (1963) we define a pressure-sensitive velocity boundary condition of the form (see (2.4))

$$u_w(y,t) = A(y)\epsilon c(0,y,t) = A(y)\epsilon^2 c_1(0,y,t)+O(\epsilon^3), \tag{3.11}$$

$$v_w(x,t) = B(x)\epsilon c(x,0,t) = B(x)\epsilon^2 c_1(x,0,t)+O(\epsilon^3). \tag{3.12}$$

After introducing (3.11) and (3.12) in (2.21) we obtain, with the help of (2.6),

$$\lambda f'(t)+\frac{\gamma+1}{8c_0^2}f'(t)f(t) + \tfrac{1}{4}(\gamma-1)\frac{\epsilon}{ab}\left\{\int_0^b\left[f\left(t+\frac{2\eta}{c_0}\sin\phi\right)+2f(t)+f\left(t-\frac{2\eta}{c_0}\sin\phi\right)\right]A(\eta)\,d\eta + \int_0^a\left[f\left(t+\frac{2\xi}{c_0}\cos\phi\right)+2f(t)+f\left(t-\frac{2\xi}{c_0}\cos\phi\right)\right]B(\xi)\,d\xi\right\} = 0. \tag{3.13}$$

It is easy to verify that all periodic solutions of (3.13) are antisymmetric about their zeros, which leads immediately to

$$\lambda = 0. \tag{3.14}$$

In this context it is important to keep in mind that the mean value of f must be zero, as required by linear acoustic theory. With (3.13) we have obtained an equation where two 'time-lag integrals' (i.e. convolution integrals) appear rather than one or more 'time-lag terms' only. Nevertheless, the numerical treatment of (3.13) is relatively simple. As a basis for the subsequent discussion we now specify the boundary conditions as follows:

$$A(y) = \delta(y-y_0), \quad B(x) = 0, \tag{3.15}$$

where δ denotes the Dirac delta function.

However, it should be pointed out that, although the specification (3.15) is convenient, it does not lead to an essential simplification as far as the numerical treatment of (3.13) is concerned, and the methods given below can readily be applied for any other choice of A and B.

The special case defined by (3.15) corresponds to the rather more realistic situation of a concentrated zone of combustion. Introducing (3.15) in (3.13) yields

$$\frac{\gamma+1}{8c_0^2} f'(t)f(t) + \tfrac{1}{4}(\gamma-1)\frac{\epsilon}{ab}\left\{f\left(t+\frac{2y_0}{c_0}\sin\phi\right)+2f(t)+f\left(t-\frac{2y_0}{c_0}\sin\phi\right)\right\} = 0. \tag{3.16}$$

Making use of the substitutions

$$t = \tau\frac{b}{c_0}\sin\phi, \quad \tau_0 = \frac{2y_0}{b}, \tag{3.17), (3.18}$$

$$f(t) = 4\frac{\gamma-1}{\gamma+1}\frac{\epsilon c_0}{a}h(\tau)\sin\phi, \tag{3.19}$$

(3.16) can be written in the form

$$h\frac{dh}{d\tau} + \tfrac{1}{4}\{h(\tau+\tau_0)+2h(\tau)+h(\tau-\tau_0)\} = 0. \tag{3.20}$$

For the special case where $\tau_0 = 0$ we obtain Chu's (1963) equation, for which we find the solutions

$$h_m(\tau+\tau_1) = -\tau + \frac{1}{m}\left\{H(\tau) + \sum_{n=1}^{\infty}\left[H\left(\tau+\frac{2n}{m}\right)+H\left(\tau-\frac{2n}{m}\right)\right]\right\}, \tag{3.21}$$

where m is a positive integer and H is defined by

$$H(\tau) = \left\{\begin{array}{rl} 1 & (\tau > 0) \\ 0 & (\tau = 0) \\ -1 & (\tau < 0) \end{array}\right\}. \tag{3.22}$$

The expressions (3.21) represent sawtooth waves, where for the fundamental mode ($m = 1$) the period is 2. The nonlinear character of (3.20) does not permit superposition of the solutions (3.21). Hence the presence of a particular mode defined by (3.21) excludes all other modes (with the same mode angle ϕ) unless a (random) disturbance favouring a different mode happens to be sufficiently strong to produce a mode transition. This very interesting property, which also involves the hysteresis character of mode transitions, can be discussed with the help of the energy equation.

If τ_0 is different from zero, (3.20) can be integrated numerically using an iteration method that is similar to that used by Mitchell (1967) to integrate (1.3), with the simplification, however, that $\lambda = 0$ and the solutions are antisymmetric about their zeros. To obtain the solution h_0 (corresponding to the fundamental mode), for example, we set $\tau_1 = 1$ and for formal reasons introduce the substitution

$$\bar{h}_0(\tau) = -h_0(\tau). \tag{3.23}$$

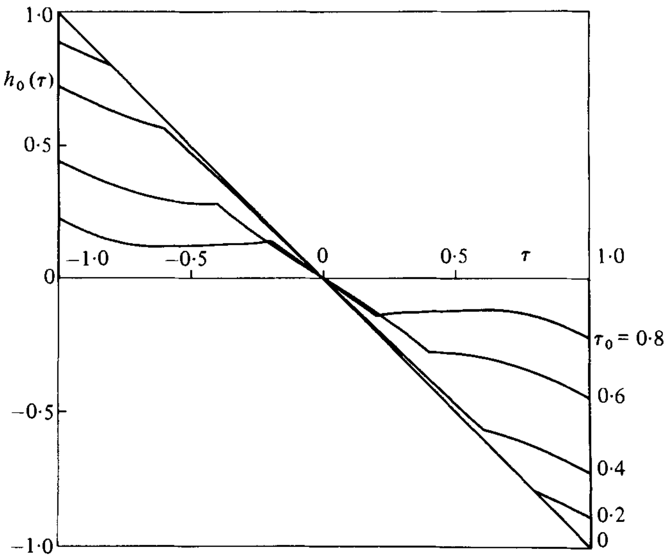

FIGURE 1. Fundamental mode according to (3.20) for different values of τ_0.

For the special case where $\tau_0 = 0$ we have from (3.21)

$$\bar{h}_0(\tau) = \tau \quad (-1 < \tau < 1), \tag{3.24}$$

where $\bar{h}_0(\tau+2) = \bar{h}_0(\tau)$.

As long as τ_0 is not too large it is convenient to use (3.24) as a first approximation $\bar{h}_{0,1}$ for the following iteration. Subsequent approximations $\bar{h}_{0,n}$ to the solution

$$\bar{h}_0 = \lim_{n\to\infty} \bar{h}_{0,n} \tag{3.25}$$

are obtained from

$$\bar{h}_{0,n+1}(\tau) = \left(\frac{1}{2}\int_0^\tau \{\bar{h}_{0,n}(\tau+\tau_0)+2\bar{h}_{0,n}(\tau)+\bar{h}_{0,n}(\tau-\tau_0)\}\,d\tau\right)^{\frac{1}{2}}, \tag{3.26}$$

for $0 \leqslant \tau < 1$, where

$$\bar{h}_{0,n}(-\tau) = -\bar{h}_{0,n}(\tau), \quad \bar{h}_{0,n}(\tau+2) = \bar{h}_{0,n}(\tau).$$

It is interesting to note that the iteration defined by (3.26) is rapidly convergent. Solutions for different values of τ_0 are shown in figure 1.

3.4. *Concluding remarks*

The analysis presented in §2 led to the wave equation (2.21), which governs nonlinear, resonant (or nearly resonant) acoustic wavefields in rectangular cavities for essentially arbitrary boundary conditions. The equation can be applied to both forced and self-excited oscillations. The investigation of self-excited oscillations based upon (2.21) in general leads to second-order equations (similar to (1.3) and (3.10)) containing terms with a time lag. Such terms appear both as a result of a 'space lag' (similar to that discussed in §3.3) and also owing to the convective character of boundary conditions (e.g. in jet-driven cavities). It has been shown that a time lag may appear as a natural outcome of the second-order analysis and that given boundary conditions can simultaneously play the role of resonant excitations functions for a certain mode and the role of non-resonant excitation functions for other modes. These two facts appear to represent the keys to a fundamental understanding of the character of the oscillations and the variety of modes appearing in self-excited resonators. It should be pointed out that, unlike t_0 in the present analysis, the time lag discussed by Mitchell *et al.* (1969) is based upon a characteristic time of combustion and represents an artifice within the framework of one-dimensional acoustics.

Finally, the problem of mode transitions should be discussed briefly. An important consequence of the nonlinearity of (2.21) is the exclusion of the simultaneous presence of oscillations with the same mode angle ϕ. Thus the selection of modes depends very much on the initial conditions. For a particular one-dimensional case this problem has been investigated by Mortell & Seymour (1973), who considered the evolution of self-sustained oscillations.

Suppose, for instance, that in a jet-driven cavity a certain mode m_1 is unstable in a mass-flow interval $Q_{1A} < Q < Q_{1B}$ of the exciting jet, and another mode m_2 is unstable in the interval $Q_{2A} < Q < Q_{2B}$, where $Q_{1A} < Q_{2A} < Q_{1B} < Q_{2B}$. Then if Q_3 ($Q_{2A} < Q_3 < Q_{1B}$) is reached by decreasing Q, mode m_2 will appear. On the other hand, if Q_3 is reached by increasing Q, m_1 appears, provided that there are no disturbances. Hence different modes may appear for exactly the same equilibrium conditions, a form of hysteresis, the appearance of which is well known in wind instruments. However, oscillations with different mode angles may appear simultaneously. This is possible because resonant excitation functions of a certain mode appear to be non-resonant for any other mode with a different mode angle, and the wave-interaction terms vanish (to second order) identically at the walls of the resonator. In other words, if P_1 and P_2 define the orthogonal projections associated with the modes m_1 and m_2 it is possible that

$$P_i[(u_{\mathrm{w}}, v_{\mathrm{w}})] = (u_{\mathrm{wR}i}, v_{\mathrm{wR}i}) \not\equiv (0, 0) \quad (i = 1, 2),$$
$$P_1 P_2[(u_{\mathrm{w}}, v_{\mathrm{w}})] \equiv (0, 0).$$

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Appendix

A.1. The pressure disturbance in the cavity can be written in the form

$$p = \frac{2\gamma}{\gamma-1}\frac{p_0}{c_0}c = \tfrac{1}{4}\gamma\frac{p_0}{c_0}\{f_{--}+f_{++}+f_{-+}+f_{+-}\}. \tag{A 1}$$

For the proof that $u_{\mathrm{wA}} = u_{\mathrm{w}} - u_{\mathrm{wR}}$ and $v_{\mathrm{wA}} = v_{\mathrm{w}} - v_{\mathrm{wR}}$, where u_{wR} and v_{wR} are given by (2.16) and (2.17), are non-resonant excitation functions we have to show that the mean rate of energy addition to the cavity is zero:

$$\left\langle \int_0^b p(0,\,y,\,t)\{u_{\mathrm{w}}(y,\,t)-u_{\mathrm{wR}}(y,\,t)\}\,dy\right\rangle + \left\langle \int_0^a p(x,\,0,\,t)\{v_{\mathrm{w}}(x,\,t)-v_{\mathrm{wR}}(x,\,t)\}\,dx\right\rangle = 0, \quad (A\ 2)$$

where the angular brackets $\langle\ \rangle$ denote time averages. When the expressions (2.16), (2.17) and (4.1) are introduced in (A 2) we obtain, after dividing (A 2) by $\gamma p_0/4c_0$,

$$\begin{aligned}\frac{\cos^2\phi}{2b}&\left\langle \int_0^b \left[\sum_{m=\pm1} f\left(t+m\frac{y}{c_0}\sin\phi\right)\right]\times\left[\sum_{k=\pm1}\sum_{l=\pm1}\right.\right.\\ &\times\left.\left.\left\{\int_0^b u_{\mathrm{w}}\left(\eta,\,t+\frac{k\eta+ly}{c_0}\sin\phi\right)d\eta+\int_0^a v_{\mathrm{w}}\left(\xi,\,t+k\frac{\xi}{c_0}\cos\phi+l\frac{y}{c_0}\sin\phi\right)d\xi\right\}\right]dy\right\rangle\\ &+\frac{\sin^2\phi}{2a}\left\langle \int_0^a \left[\sum_{m=\pm1} f\left(t+m\frac{x}{c_0}\cos\phi\right)\right]\left[\sum_{k=\pm1}\sum_{l=\pm1}\right.\right.\\ &\times\left.\left.\left\{\int_0^b u_{\mathrm{w}}\left(\eta,\,t+k\frac{x}{c_0}\cos\phi+l\frac{\eta}{c_0}\sin\phi\right)d\eta+\int_0^a v_{\mathrm{w}}\left(\xi,\,t+\frac{kx+l\xi}{c_0}\cos\phi\right)d\xi\right\}\right]dy\right\rangle\\ =&\left\langle \int_0^b \left[\sum_{m=\pm1} f\left(t+m\frac{y}{c_0}\sin\phi\right)\right]u_{\mathrm{w}}(y,\,t)\,dy+\int_0^a\left[\sum_{m=\pm1} f\left(t+m\frac{x}{c_0}\cos\phi\right)\right]v_{\mathrm{w}}(x,t)\,dx\right\rangle.\end{aligned} \quad (A\ 3)$$

We consider two typical terms that are obtained after multiplying the sums on the left-hand side of (A 3):

$$\frac{\cos^2\phi}{2b}\left\langle \int_0^b f\left(t\pm\frac{y}{c_0}\sin\phi\right)\int_0^b u_{\mathrm{w}}\left(\eta,\,t+\frac{\eta}{c_0}\sin\phi+\frac{y}{c_0}\sin\phi\right)d\eta\,dy\right\rangle. \quad (A\ 4)$$

After a suitable shift of the origin on the time axis, the first term (upper sign in the argument of f) becomes

$$\frac{\cos^2\phi}{2b}\left\langle \int_0^b f(t)\int_0^b u_{\mathrm{w}}\left(\eta,\,t+\frac{\eta}{c_0}\sin\phi\right)d\eta\,dy\right\rangle = \tfrac{1}{2}\cos^2\phi\left\langle \int_0^b f(t)\,u_{\mathrm{w}}\left(\eta,\,t+\frac{\eta}{c_0}\sin\phi\right)d\eta\right\rangle. \quad (A\ 5)$$

After renaming the dummy variable η as y and a further shift of the origin on the time axis we obtain

$$\tfrac{1}{2}\cos^2\phi\left\langle \int_0^b f\left(t-\frac{y}{c_0}\sin\phi\right)u_{\mathrm{w}}(y,t)\,dy\right\rangle. \quad (A\ 6)$$

The second term defined by (A 4) (lower sign in the argument of f) becomes, after a suitable shift of the origin on the time axis,

$$\frac{\cos^2\phi}{2b}\left\langle \int_0^b f\left(t-\frac{2y}{c_0}\cos\phi\right)\int_0^b u_{\mathrm{w}}\left(\eta,\,t+\frac{\eta}{c_0}\sin\phi\right)d\eta\,dy\right\rangle. \quad (A\ 7)$$

Making use of the periodicity condition

$$f\left(t+\frac{b}{c_0}\cos\phi\right)=f\left(t-\frac{b}{c_0}\cos\phi\right) \quad (A\ 8)$$

and the mean-value condition (required by linear acoustic theory)

$$\langle f(t)\rangle = 0, \quad (A\ 9)$$

we can easily show that the term (A 7) disappears identically.

All other terms on the left-hand side of (A 3) can be discussed by analogy to the terms (A 4). After this it is easy to show that (A 3) is an identity.

A.2. Without an essential loss of generality we restrict the consideration to exactly resonant oscillations (i.e. $\lambda = 0$). Making use of $A_{II} \equiv 0$ and $A_I(0, y, t) = 0$, we obtain from (2.7) at $x = 0$

$$\epsilon^2 u_2(0, y, t) = -\frac{\gamma+1}{32c_0^2} a \cos^2\phi \sum_{l=\pm 1} \frac{\partial}{\partial t} f^2\left(t + l\frac{y}{c_0}\sin\phi\right), \tag{A 10}$$

and at $x = a$, with the help of $A_I(a, y, t) = 0$ and the periodicity condition $f(t+(a/c_0)\cos\phi) = f(t-(a/c_0)\cos\phi)$,

$$\epsilon^2 u_2(a, y, t) = 0. \tag{A 11}$$

From (2.8) we obtain at $y = 0$

$$\epsilon^2 v_2(x, 0, t) = -\frac{\gamma+1}{32c_0^2} b \sin^2\phi \sum_{k=\pm 1} \frac{\partial}{\partial t} f^2\left(t + k\frac{x}{c_0}\cos\phi\right), \tag{A 12}$$

and at $y = b$

$$\epsilon^2 v_2(x, b, t) = 0, \tag{A 13}$$

where use has been made of $B_{II} \equiv 0$, $B_I(x, 0, t) = B_I(x, a, t) = 0$ and the periodicity condition $f(t+(b/c_0)\sin\phi) = f(t-(b/c_0)\sin\phi)$. Both (2.16) and (A 10) represent superpositions of two waves that run in the positive and negative direction along the y-axis with the speeds $\pm c_0/\sin\phi$ respectively. As the two characteristic variables $t \pm y\sin\phi/c_0$ of the two wave functions are independent, equating (2.16) and (A 10) yields two equations. After substitution of the expression (2.16) and (A 10) in

$$u_{wR}(y, t) - \epsilon^2 u_2(0, y, t) = 0, \tag{A 14}$$

we obtain

$$\sum_{l=\pm 1} \tfrac{1}{2} a \cos^2\phi \left\{\frac{\gamma+1}{8c_0^2} f(\tau_l) f'(\tau_l) + \frac{1}{ab} \sum_{k=\pm 1}\left[\int_0^b u_w\left(\eta, \tau_l + \frac{k\eta}{c_0}\sin\phi\right) d\eta \right.\right.$$
$$\left.\left. + \int_0^a v_w\left(\xi, \tau_l + \frac{k\xi}{c_0}\cos\phi\right) d\xi\right]\right\} = 0, \tag{A 15}$$

where

$$\tau_{\pm 1} = t \pm y\sin\phi/c_0 \tag{A 16}$$

are the independent characteristic variables. Thus, two equations for f are obtained by putting the two expressions for $l = \pm 1$ within the curly brackets of (A 15) (separately) equal to zero.

It is easy to see that for $\lambda = 0$ both equations are identical with (2.21). The same is true for the two equations which are obtained from equating the wave functions (2.17) and (A 12). Thus the proof is given that the boundary conditions (2.16) and (2.17) can be satisfied by the reduced second-order boundary conditions, and that (2.16) and (2.17) also represent resonant excitation functions according to the definitions (2.13) and the subsequent remarks.

REFERENCES

CHESTER, W. 1964 Resonant oscillations in closed tubes. *J. Fluid Mech.* **18**, 44.

CROCCO, L. & CHENG, S. I. 1956 Theory of combustion instability in liquid propellant rocket motors. *AGARDograph* no. 8.

CHU, B. T. 1963 Analysis of a self-sustained nonlinear vibration in a pipe containing a heater. *Phys. Fluids* **6**, 1638.

KELLER, J. J. 1976*a* Resonant oscillations in closed tubes: the solution of Chester's equation. *J. Fluid Mech.* **77**, 279.

KELLER, J. J. 1976*b* Third order resonances in closed tubes. *Z. angew. Math. Phys.* **27**, 303.

KELLER, J. J. 1977 Resonant oscillations in open tubes. *Z. angew. Math. Phys.* **28**, 237.

KELLER, J. J. 1978*a* Nonlinear self-excited acoustic oscillations. *Z. angew. Math. Phys.* **29**, 934.

KELLER, J. J. 1978*b* A note on nonlinear acoustic resonances in rectangular cavities. *J. Fluid Mech.* **87**, 299.

MITCHELL, C. E. 1967 Axial mode shock wave combustion instability in liquid propellant rocket engines. *Princeton University Dept. of Aerospace and Mech. Sci. Tech. Rep.* no. 798 (*NASA* CR-72259).

MITCHELL, C. E., CROCCO, L. & SIRIGNANO, W. A. 1969 Nonlinear longitudinal instability in rocket motors with concentrated combustion. *Combust. Sci. Tech.* **1**, 35.

MORTELL, M. P. & SEYMOUR, B. R. 1973 The evolution of a self-sustained oscillation in a nonlinear continuous system. *Trans. A.S.M.E.* E: *J. Appl. Mech.* **40**, 53.